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THE STABILITY OF A CLASS OF QUASI-AUTONOMOUS PERIODIC SYSTEMS WITH INTERNAL RESONANCE†

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The problem of the stability of the periodic motion of a multi-dimensional periodic system with a small parameter is considered. The system is autonomous when that parameter is zero. The critical case when the characteristic indices consist of a zero and N pairs of purely imaginary numbers is investigated on the assumption that the system is reversible (t-invariant) in the Birkhoff sense. It is shown that when there is no parametric resonance the characteristic indices of the original system are identical with the characteristic indices of the corresponding autonomous system to first-order in the small parameter if there are no multiplicities in the autonomous system. Cases of third- and fourth-order internal resonance are then considered. Here the system can be unstable and a non-linear analysis is required. Necessary and sufficient conditions for stability are found, and the influence of small periodic terms on the stability is established.

The results obtained are used to investigate the periodic piecewise-orbital motion of a geostationary satellite with a small reactive thrust that allows it to hover over any point of the Earth's surface.

1. CONSIDER the system of equations of perturbed motion

$$x^{\cdot}=X(x,t,\varepsilon),\quad x\in R^{2N+1} \tag{1.1}$$

$$X(x,t,\varepsilon)=\left[A_0+\sum_{k=1}^{\infty}\varepsilon^k A_k(t)\right]x+X_0(x)+\sum_{k=1}^{\infty}\varepsilon^k X_k(x,t)$$

where $x=(x_1,\ \ldots,\ x_{2N+1})$ is the phase-variable vector, $X_0(x)$ is an analytic vector-function of x containing no terms that are of lower than the second order in x, $X_k(x,\ t)$ are ω-periodic functions of time t, analytic in a neighbourhood of zero in x, and containing no terms that are of lower than the second order in x, A_0 and $A_k(x)$ are respectively a constant and ω-periodic $(2N+1)\times(2N+1)$ matrices such that the linear part of system (1.1) has only one zero and N pairs of purely imaginary characteristic indices $\pm\lambda_s$ $(\lambda_s^2<0;\ s=1,\ \ldots,\ N)$ and ε is a small parameter. The values of the matrix elements of $A_k(t)$ averaged over a period are zero, which obviously does not cause any loss of generality. Furthermore, we shall assume that system (1.1) is reversible [1, 2] (or t-invariant [3]), i.e. the following identity exists

$$MX(x,t,\varepsilon)+X(Mx,-t,\varepsilon)\equiv 0,\quad M^2=E$$

(E is the unit matrix).

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In analysing the stability of the trivial solution of system (1.1) when $\varepsilon = 0$ it was shown [4, 5] that this system can, in particular, describe the perturbed piecewise-orbital motion of a geostationary satellite which uses a small thrust to ensure motion along a circular orbit of arbitrary diameter. One can show that the investigation of the stability of periodic motions appearing in a neighbourhood of the relative equilibrium positions considered [4] lead to equations of perturbed motion of the form (1.1), in which the role of the small parameter ε is played by the eccentricity of the elliptical orbit of the centre of mass of the satellite.

It is known [6] that the question of the stability of the trivial solution of system (1.1) reduces to the question of the stability of an autonomous system with one zero and N pairs of purely imaginary roots considered in [5].

For a system containing a small parameter it is interesting to clarify the role of this parameter in the solution of the stability problem.

We first transform the linear part of system (1.1), reducing it to autonomous form, and to this end introduce new variables $z = (z_1, \ldots, z_{2N+1})$ with the formulae

$$x = [B_0 + \varepsilon B_1(t) + \ldots]z \tag{1.2}$$

where B_0 is a constant and $B_1(t)$ is a matrix ω-periodic in t.

We require that in the new variables the system has the form

$$z^{\cdot} = [\Lambda_0 + \varepsilon\Lambda_1 + \ldots]z \tag{1.3}$$

where Λ_0 and Λ_1 are constant diagonal matrices.

Differentiating (1.2) with respect to time and using (1.1), we obtain the following systems of equations for the jth columns of the matrices B_0 and B_1

$$(A_0 - \lambda_{0j}E)B_{0j} = 0 \quad (j = 1, \ldots, 2N+1) \tag{1.4}$$

$$B_{1j}^{\cdot} - (A_0 - \lambda_{0j}E)B_{1j} = (A_1 - \lambda_{1j}E)B_{0j} \tag{1.5}$$

where λ_{0j} and λ_{1j} are elements of the matrices Λ_0 and λ_1, respectively. (It follows from (1.4) that the λ_{0j} are the eigenvalues of the matrix A_0.)

We consider the question of the existence of an ω-periodic solution to system (1.5). All the roots of the characteristic equation of each of these systems is found from the formula

$$\kappa_{js} = i(\operatorname{Im}\lambda_{0s} - \operatorname{Im}\lambda_{0j}) \quad (s, j = 1, \ldots, 2N+1) \tag{1.6}$$

Introducing the matrix $e_j(t) = \operatorname{diag}(e^{\kappa_{j1}t}, \ldots e^{\kappa_{j2N+1}t})$, we consider the normalized fundamental matrix

$$B_{1j}^{n}(t) = B_0 e_j(t) B_0^{-1} \tag{1.7}$$

We take a particular solution of the inhomogeneous system (1.5) in Cauchy form

$$B_{1j}^{*}(t) = \int_{n}^{t} B_{1j}^{n}(t-\tau)[A_1(\tau) - \lambda_{1j}E]B_{0j}d\tau \tag{1.8}$$

and choose the arbitrary constants $C_j = (C_{j1}, \ldots, C_{jN+1})^T$ of the general solution from the periodicity condition which leads to the system of equations

$$[B_{1j}^{n}(\omega) - E]C_j + B_{1j}^{*}(\omega) = 0 \tag{1.9}$$

Since, as follows from (1.6), the characteristic equation of each of the systems (1.5) must have one (and only one) zero root, for system (1.9) we have $\det[B_{1j}^{n}(\omega) = E] = 0$, and consequently

$$\operatorname{rank}[B_{1j}^n(\omega)-E]=2N \tag{1.10}$$

if the equality

$$\kappa_{sj}\,\omega=2ik\pi \tag{1.11}$$

is not satisfied by any natural k, and this we shall assume below. (For a second-order system with a pair of imaginary eigenvalues this relation obviously represents the condition of parametric resonance.)

It follows from (1.10) that system (1.9) will either have no solutions or an infinite number of them. In the latter case the matrix $B_{1j}^*(\omega)$ should satisfy the condition

$$\operatorname{rank}\{[B_{1j}^n(\omega)-E],\ -B_{1j}^*(\omega)\}=2N \tag{1.12}$$

Condition (1.12) can be satisfied by putting $\lambda_{1j}=0$ in (1.8).

Indeed, taking (1.7) and (1.8) into account, we can write

$$B_{1j}^*(\omega)=\int_0^\omega B_{1j}^n(\omega-\tau)A_1(\tau)B_{0j}d\tau=\int_0^\omega B_{1j}^n(\omega-\tau)A_{1j}(\tau)d\tau \tag{1.13}$$

where the elements of the column matrix $A_{1j}(\tau)$ are ω-periodic functions of τ that are expandable in a Fourier series. This enables us to represent the final integral in (1.13) in the form

$$B_{1j}^*(\omega)=B_0e(\omega-\tau)B_0^{-1}A_{1j}^*(\tau)\Big|_0^\omega=-[B_{1j}^n(\omega)-E]A_{1j}^*(\omega)$$

$$A_{1j}^*(\omega)=A_{1j}^*(0)$$

From this it is clear that the last column of the widened matrix appearing in (1.12) is a linear combination of all the other columns, and consequently condition (1.12) is satisfied when $\lambda_{1j}=0$.

An important conclusion has therefore been obtained to the effect that in the absence from the system of the commensurability condition (1.11) the characteristic indices of the original system are identical to first order of accuracy in ε with the eigenvalues of the matrix A_0. This conclusion can also be obtained by more complicated arguments [3].

2. On the basis of the above, system (1.1) can be represented in the form

$$\begin{aligned}
&y^\cdot=\sum_{m=2}^\infty Y^{(m)}+Y, \quad u^\cdot=\Lambda_*u+\sum_{m=2}^\infty U^{(m)}+U,\\
&\bar u^\cdot=-\Lambda_*\bar u+\sum_{m=2}^\infty \bar U^{(m)}+\bar U\\
&\Lambda_*=\operatorname{diag}(i\lambda_{01},\dots,i\lambda_{0N})\\
&u=(u_1,\dots,u_N), \quad \bar u=(\bar u_1,\dots,\bar u_N)
\end{aligned} \tag{2.1}$$

where u and $\bar u$ are complex-conjugate variables, Y, U and $\bar U$ are analytic functions of y, u, $\bar u$, t and ε containing powers of ε not lower than two and $Y^{(m)}$, $U^{(m)}$ and $\bar U^{(m)}$ are mth order forms with ω-periodic coefficients representable in the form

$$Y^{(m)}=Y_0^{(m)}(y,u,\bar u)+\varepsilon Y_1^{(m)}(y,u,\bar u,t), \quad U^{(m)}=U_0^{(m)}(y,u,\bar u)+\varepsilon U_1^{(m)}(y,u,\bar u,t) \tag{2.2}$$

If system (2.1) has the automorphism $t \to -t$, $u \to \bar{u}$, $\bar{u} \to u$, $y \to y$ (which from now on we assume), then by a suitable choice of normalizing transformation these forms will only have purely imaginary coefficients [4].

It is known [7] that the systems under consideration are formally stable if the frequencies do not possess any $|p|$th order resonant relations of the form

$$\begin{gathered} \langle p,\lambda_0\rangle = 2\pi\omega^{-1}q; \quad q = 0,\pm1,\pm2,\ldots \\ |p| = p_1+\ldots+p_n \geqslant 3, \quad 1 \leqslant n \leqslant N, \quad p = (p_1,\ldots,p_n) \\ \lambda_0 = (\lambda_{01},\ldots,\lambda_{0n}) \end{gathered} \tag{2.3}$$

where p is an integer vector with mutually-prime positive components.

We will consider those resonant cases in which the stability question is resolved by the first non-linear terms, i.e. third- and fourth-order resonances; in each of these cases the solution of the stability problem very largely depends on whether or not the right-hand side of (2.3) vanishes.

We will first consider the case when $|p| = 3$ and $q \neq 0$. Performing a non-linear normalization of system (2.1) using the well-known procedure [8], we obtain in polar coordinates r_s, θ_s the following model system (containing only the first non-linear terms)

$$\begin{gathered} y^{\cdot} = 0, \quad r_s^{\cdot} = 2\varepsilon r^{p/2} b_s \sin\theta \quad (s = 1,\ldots,n) \\ \theta^{\cdot} = \sum_{s=1}^{n} p_s[(\beta_s^0 + \varepsilon\beta_s)y + \varepsilon b_s r^{p/2} r_s^{-1}\cos\theta] \\ r_\alpha^{\cdot} = 0, \quad \theta_\alpha^{\cdot} = \lambda_{0\alpha} + (\beta_\alpha^0 + \varepsilon\beta_\alpha)y \quad (\alpha = n+1,\ldots,N) \\ \theta = p_1\theta_1+\ldots+p_n\theta_n, \quad r^{p/2} = \prod_{j=1}^{n} r_j^{p_j/2}, \quad |p| = 3 \end{gathered} \tag{2.4}$$

Here b_s, β_s^0 and β_s are constant coefficients whose expressions in terms of the coefficients of the forms (2.2) are known [8].

The model system that has been obtained is a special case of the systems considered in [4, 5], where it was shown that the problem of the stability of the trivial solution is governed exclusively by the resonance coefficients b_s, which appear only as a consequence of the presence of small periodic terms in the original system. Necessary and sufficient conditions for stability were derived which have to be satisfied by these conditions.

For $|p| = 3$ and $q = 0$ the model system differs from (2.4) only in the equations for the resonant variables r_s, which take the form

$$r_s^{\cdot} = 2(b_s^0 + \varepsilon b_s)r^{p/2}\sin\theta \quad (s = 1,\ldots,n) \tag{2.5}$$

where the constant coefficients b_s^0 are computed in terms of the constant coefficients of the forms $Y_0^{(m)}$, $U_0^{(m)}$ and $\bar{U}_0^{(m)}$ of system (2.2).

The problem of the stability of the trivial solution of the model system obtained is solved only by the group of equations (2.5) [4, 5]: the necessary and sufficient condition for stability is the presence of a sign change in the series of coefficients $a_s = b_s^0 + \varepsilon b_s$ of system (2.5). But because for sufficiently small values of ε the signs of a_s are the same as the signs of b_s^0, in the case under consideration the solution of the problem of the stability of the original system can be obtained while ignoring its periodic part.

We now consider fourth-order resonance. For $q \neq 0$ the model system, containing terms up to the third-order inclusive, can be written as

$$
\begin{aligned}
&y^{\cdot}=0, \quad r_s^{\cdot}=2\varepsilon b_s r^{p/2}\sin\theta \quad (s=1,\ldots,n)\\
&\theta^{\cdot}=\sum_{s=1}^{n} p_s[(\beta_{1s}^0+\varepsilon\beta_{1s})y+(\beta_{2s}^0+\varepsilon\beta_{2s})y^2+\\
&+\sum_{j=1}^{N}(\gamma_{sj}^0+\varepsilon\gamma_{sj})r_j+\varepsilon b_s r^{p/2} r_s^{-1}\cos\theta]\\
&r_\alpha^{\cdot}=0, \quad \theta_\alpha^{\cdot}=\lambda_{\alpha 0}+(\beta_{1\alpha}^0+\varepsilon\beta_{1\alpha})y+(\beta_{2\alpha}^0+\varepsilon\beta_{2\alpha})y^2+\\
&+\sum_{j=1}^{N}(\gamma_{\alpha j}^0+\varepsilon\gamma_{\alpha j})r_j \quad (\alpha=n+1,\ldots,N)\\
&\theta=p_1\theta_1+\ldots+p_n\theta_n, \quad |p|=4
\end{aligned}
\tag{2.6}
$$

Thus, in this case the original periodic system reduces to an autonomous system which is a special case of the systems considered in [5]. According to results obtained in [5], a necessary and sufficient condition for the stability of the trivial solution of system (2.6) in the non-degenerate case (all $b_s \neq 0$) is either the presence of a pair of coefficients b_ν and b_μ of opposite sign, or if the series b_s does not change sign, the satisfaction of the inequality

$$
\left|\sum_{s,j=1}^{n} p_s(\gamma_{sj}^0+\varepsilon\gamma_{sj})\varepsilon b_s\right| > \prod_{j=1}^{n}\left|\varepsilon b_j\right|^{p_j/2}
\tag{2.7}
$$

Obviously, in the case of fourth-order resonance ($|p|=4$) for sufficiently small values of ε inequality (2.7) will always be satisfied, and hence the trivial solution of system (2.6) in the $|p|=4$, $q\neq 0$ case is always stable.

Suppose now that with a fourth-order resonance in (2.3) we have $q=0$. Then the normalized model system will differ from (2.6) only in the equations for the resonant variables r_s. These equations will have the same form as (2.5), but with $|p|=4$. We see that the signs of all the $a_s=b_s^0+\varepsilon b_s$ and $c_{sj}=\gamma_{sj}^0+\varepsilon\gamma_{sj}$ for sufficiently small values of ε will be the same as the signs of b_s^0 and γ_{sj}^0, and consequently, in the solution of the stability problem for this case the periodic terms in the original system can be ignored. The stability conditions for the autonomous system obtained are given by a theorem from [5]. Thus, based on the above one can formulate the following theorem.

Theorem. Suppose that in system (1.1) there is an internal resonance (2.3) of third or fourth order with $q=0$. Then, when solving the stability problem for sufficiently small ε, one can consider the corresponding autonomous system obtained from (1.1) with $\varepsilon=0$ instead of the original periodic system. For the case when $q\neq 0$ the stability of the trivial solution of system (1.1) with third-order resonance is governed exclusively by the periodic terms, no matter how small ε is; at fourth-order resonance and $q\neq 0$ the stability of the trivial solution of system (1.1) is still preserved when non-linear terms of up to third order inclusive are taken into account.

3. The analysis that has been performed enables us to generalize previous investigations [4, 8] into the stability of piecewise-orbital motion of a geostationary artificial satellite that is suspended over any point on the Earth's surface as a result of a small accelerative thrust w that is constant in modulus. We shall show that in a small neighbourhood of the stable stationary motions found in [4] that are relative equilibria of the satellite in a uniformly rotating system of coordinates comoving with the Earth, stable periodic motions exist with periods close to the Earth's rotational period, and with the same set of unstable resonant regimes.

To this end the equations of motion of such a satellite represented, as in [4, 8], as a body of variable mass with a rigid surface, are written in a Cartesian system of coordinates xyz rotating with angular velocity ω, whose origin is located at the centre of the Earth, with the z axis about which the rotation takes place directed along the Earth's axis of rotation. We take ω to be equal to the angular velocity of the satellite along an elliptic Keplerian orbit with semi-axis a equal to the radius of the orbit of a stationary equatorial satellite and an arbitrary, but sufficiently small eccentricity e, so that

$$\omega \equiv \frac{dv}{dt} = \sqrt{\frac{\mu}{a^3}}\frac{(1+e\cos v)^2}{(1-e^2)^{3/2}} \tag{3.1}$$

where v is the true anomaly of the elliptical orbit considered, and μ is the gravitational parameter of the Earth.

Taking into account the smallness of e instead of (3.1) we shall use in what follows the approximate expression

$$dv/dt = \omega_e(1+2e\cos v) \tag{3.2}$$

where ω_e is the angular velocity of the Earth about its axis.

Then using the notation and assumptions of [18] we obtain the following equations for the piecewise-orbital motion of a satellite with a small thrust of constant magnitude, whose vector is stationary in a system of coordinates attached to the satellite and which passes through its centre of mass

$$\begin{aligned}
&R^{\cdot\cdot} - R(\theta^{\cdot 2}\cos\varphi + \varphi^{\cdot}) - R\omega^2\cos^2\varphi - 2R\omega\theta^{\cdot}\cos^2\varphi = w\sum_{i=1}^{3}\gamma_i\sigma_i + \frac{1}{m}\frac{\partial U}{\partial R}\\
&\frac{d}{dt}(R^2\varphi^{\cdot}) + \frac{1}{2}R(R\theta^{\cdot 2}\sin^2\varphi + \omega^2\sin 2\varphi) + \omega R^2\theta^{\cdot}\sin 2\varphi = Rw\sum_{i=1}^{3}\beta_i\sigma_i\\
&\frac{d}{dt}(R^2\theta^{\cdot}\cos^2\varphi) + R^2\omega^{\cdot}\cos^2\varphi + R\omega(2R^{\cdot}\cos\varphi - R\varphi^{\cdot}\sin 2\varphi) = Rw\cos\varphi\sum_{i=1}^{3}\alpha_i\sigma_i\\
&\beta^{\cdot} = r\beta_2 - q\beta_3 - [(\theta^{\cdot}+\omega)\alpha_1\sin\varphi + \gamma_1\varphi^{\cdot}]\\
&\gamma_1^{\cdot} = r\gamma_2 - q\gamma_3 + [(\theta^{\cdot}+\omega)\alpha_1\cos\varphi + \beta_1\varphi^{\cdot}]\\
&\gamma_2^{\cdot} = p\gamma_3 - r\gamma_1 + [(\theta^{\cdot}+\omega)\alpha_2\cos\varphi + \beta_2\varphi^{\cdot}]\\
&Ap^{\cdot} + (C-B)qr = \Omega^2(C-B)\gamma_2\gamma_3 \quad (ABC,\ pqr,\ \gamma_1\gamma_2\gamma_3)\\
&\Omega^2 = 3\mu/R^3
\end{aligned} \tag{3.3}$$

Here R, φ and θ are, respectively, the distance to the centre of the Earth, and the latitude and longitude of the centre of mass of the satellite in the previously mentioned rotating system of coordinates, α_i, β_i and γ_i are the direction cosines of the principal axes of inertia of the satellite with respect to the introduced spherical (orbital) system, σ_i are the direction cosines of the vector w with respect to the comoving axes, p, q and r are the projections of the absolute angular velocity of the satellite onto the comoving axes, and A, B and C are the squares of the radii of inertia with respect to the principal axes, assumed to be constant; differentiation with respect to time t is denoted by an overdot.

Changing from the time t to a new independent variable v and putting

$$R = y_1, \quad \varphi = y_2, \quad d\theta/dv \equiv \theta' = y_3, \quad R' = y_4, \quad \varphi' = y_5$$

$$p = y_6, \quad q = y_7, \quad r = y_8, \quad \beta_1 = y_9, \quad \gamma_1 = y_{10}, \quad \gamma_{11} = y_{11}, \quad \theta = y_{12}$$

instead of (3.3) we obtain the following system in which non-linear terms in the small parameter e are neglected

$$y_s' = Y_s(e, y_1, \dots, y_{11}, \nu) \quad (s = 1, \dots, 11) \quad y_{12}' = y_3 \tag{3.4}$$

Here

$$\begin{aligned}
&Y_1 = y_4, \quad Y_2 = y_5, \quad Y_3 = 2ey_3 \sin\nu - 2y_3y_4 / y_1 + 2y_3y_5 \operatorname{tg} y_2 - \\
&-2y_4 / y_1 + 2y_5 \operatorname{tg} y_2 + 2e\sin\nu + w\sum_{i=1}^{3} \alpha_i\sigma_i / (\nu'^2 y_1), \\
&Y_4 = 2ey_4 \sin\nu + y_1\cos^2 y_2 + (y_3^2\cos^2 y_2 + y_5^2)y_1 + 2y_1y_3\cos^2 y_2 + \\
&+\left(w\sum_{i=1}^{3}\gamma_i\sigma_i - \mu / y_1^2\right)/\nu'^2, \quad Y_5 = 2ey_5\sin\nu - 2y_4y_5 / y_1 - \\
&-\tfrac{1}{2}y_3^2\sin 2y_2 - \sin y_2\cos y_2 - y_3\sin 2y_2 + w\sum_{i=1}^{3}\beta_i\sigma_i / (\nu'^2 y_1) \\
&Y_6 = (B - C)(y_7y_8 - 3\mu\gamma_2\gamma_3 / y_1^3) / (A\nu') \\
&Y_7 = (C - A)(y_6y_8 - 3\mu\gamma_1\gamma_3 / y_1^3)(B\nu') \\
&Y_8 = (A - B)(y_6y_7 - 3\mu\gamma_1\gamma_2 / y_1^3) / (C\nu') \\
&Y_9 = (\beta_2y_8 - \beta_3y_7) / \nu' - (\alpha_1y_3\sin y_2 + y_5y_{10}) - (\omega_e\alpha_1\sin y_2) / \nu' \\
&Y_{10} = (y_8y_{11} - \gamma_3y_7) / \nu' + \alpha_1y_3\cos y_2 + y_5y_9 + (\omega_e\alpha_1\cos y_2) / \nu' \\
&Y_{11} = (y_6\gamma_3 - y_8y_{10}) / \nu' + \alpha_2y_3\cos y_2 + \beta_2y_5 + (\omega_e\alpha_2\cos y_2) / \nu'
\end{aligned} \tag{3.5}$$

where in accordance with approximation (3.2) one must take

$$1/\nu' = \omega_e^{-1}(1 - 2e\cos\nu), \quad 1/\nu'^2 = \omega_e^{-2}(1 - 4e\cos\nu)$$

For $e = 0$ system (3.4) has the particular solution

$$\begin{aligned}
&y_1 = y_{10}, \quad y_2 = y_{20}, \quad y_3 = y_4 = y_5 = y_6 = y_9 = y_{10} = \alpha_2 = \alpha_3 = \sigma_1 = 0, \\
&y_7 = \omega_e, \quad \beta_2 = \gamma_3 = \cos\lambda, \quad \gamma_2 = -\beta_3 = \sin\lambda
\end{aligned} \tag{3.6}$$

the orbital stability of which was investigated in [4], (the final equation of system (3.4) was neglected because the variable y_{12} does not enter into the fundamental system governing the orbital stability of the satellite), and it was shown that the domain of orbital stability of the stationary motion (3.6) and the unstable resonant sets are almost identical with those which were constructed [8] using a non-central model of the gravitational field of the Earth.

Because the right-hand side of system (3.4) is an analytic function of the small parameter ε and is 2π-periodic in ν, then by a theorem of Poincaré [9] it can have a 2π-periodic solution, analytic in e and reducing to solution (3.6) when $e = 0$ if the characteristic equation of the varied system of equations about $e = 0$ does not have roots of the form $\pm ki$, $(k = 0, 1, 2, \dots)$; in the opposite case the right-hand sides of the corresponding inhomogeneous system should satisfy given conditions.

Representing the required periodic solution in the form

$$z(\nu) = y^* + eu(\nu) + \ldots$$
$$z = (z_1, \ldots, z_{11}), \quad u = (u_1, \ldots, u_{11})$$

where $y^* = (y_1^*, \ldots, y_{11})$ is a particular solution of (3.6), and putting $Y_s = Y_s^0(y) + eY_s^1(y, \nu)$ in (3.4), we obtain for $u_s(\nu)$ the system of equations

$$u_s' = \sum_{\alpha=1}^{11} \left(\frac{\partial Y_s^0}{\partial y_\alpha}\right)_* u_\alpha + Y_s^1(y_*, \nu) \quad (s = 1, \ldots, 11) \tag{3.7}$$

in which the constant matrix $\{\partial Y_s^0/\partial y_\alpha\}_*$ (the asterisk denoting the result of substituting the values of (3.6) in place of the y_α) is the matrix of the system of variational equations for the particular solution (3.6), the stability of which was investigated in [4], where it was shown that for all values of the system parameters this matrix can only have one zero eigenvalue. Thus system (3.7) can have a 2π-periodic solution only in the case when the functions $Y_s^1(y_*, \nu)$ satisfy the conditions

$$\int_0^{2\pi} Y_s^1(y_*, \nu) d\nu = 0 \tag{3.8}$$

and the remaining eigenvalues do not include purely imaginary values equal to $\pm i$ [9]. From (3.5) it is clear that condition (3.8) is satisfied and, consequently, system (3.7) will have a 2π-periodic solution if one excludes from consideration the set of parameter values for which the above pair of imaginary roots occurs, and this we shall assume to be true in the following.

One can verify that the equations of perturbed motion for the periodic motion under investigation will have the form of system (1.1). Thus, putting $x_s = y_s - z_s$, for the equations of the first approximation we obtain

$$x_s' = \sum_{\alpha=1}^{11} \left\{ \left(\frac{\partial Y_s^0}{\partial y_\alpha}\right)_* + e\left[\sum_{\beta=1}^{11} \left(\frac{\partial^2 Y_s^0}{\partial y_\beta \partial y_\alpha}\right)_* u_\beta + \left(\frac{\partial Y_s^1}{\partial y_\alpha}\right)_*\right] + \ldots \right\} x_\alpha \tag{3.9}$$

where the terms that have not been written out are of higher than the first order in the small parameter e.

As we see, for $e = 0$ Eq. (3.9) reduces to the system considered in [4] and admits of the linear automorphism

$$x_1 \to x_1, \quad x_2 \to x_2, \quad x_3 \to x_3, \quad x_4 \to x_4$$
$$x_5 \to -x_5, \quad x_6 \to -x_6, \quad x_7 \to x_7, \quad x_8 \to x_8$$
$$x_9 \to -x_9, \quad x_{10} \to x_{10}, \quad x_{11} \to -x_{11}, \quad \nu \to -\nu$$

which, as can be verified, is also the case for system (3.9). It too is therefore reversible. From this and from Sec. 1 it follows that the characteristic indices of system (3.9) will differ from the eigenvalues of the matrix $\{\partial Y_s^0/\partial y_\alpha\}_*$ governing the stability of the stationary motion (3.6) by terms of the order of e^2, and hence the domain of stability of the periodic motion considered in the space of parameters B/A, C/S and φ, will, for sufficiently small values of e, be almost identical with the domain of stability of the above stationary motion constructed in [8]. From the reversibility of system (3.9) it also follows that this domain will at the same time be the domain of complete stability in the sense of Birkhoff [1], i.e. stability to any finite order.

Based on the theorem proved in Sec. 2, one can also conclude that the instability of stationary motion at third-order resonance discovered in [4, 8] corresponds to instability for the same values of the parameters in the periodic motion under consideration.

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